NSW EDUCATION STANDARDS AUTHORITY

2017 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7–14)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which polynomial is a factor of $x^3 - 5x^2 + 11x - 10$?

A. $x - 2$

B. $x + 2$

C. $11x - 10$

D. $x^2 - 5x + 11$

2 It is given that $\log_a 8 = 1.893$, correct to 3 decimal places.

What is the value of $\log_a 4$, correct to 2 decimal places?

A. 0.95

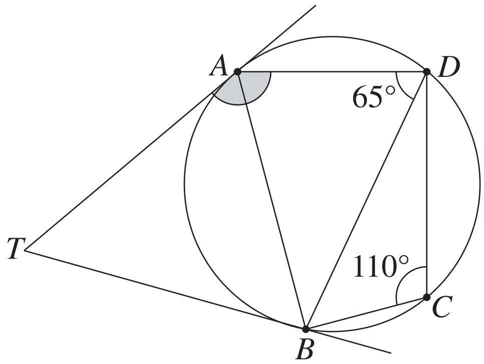
B. 1.26

C. 1.53

D. 2.84

3 The points A, B, C and D lie on a circle and the tangents at A and B meet at T, as shown in the diagram.

The angles BDA and BCD are 65° and 110° respectively.



What is the value of $\angle TAD$?

A. 130°

B. 135°

C. 155°

D. 175°

4 What is the value of $\tan\alpha$ when the expression $2\sin x - \cos x$ is written in the form $\sqrt{5}\sin(x-\alpha)$?

A. -2

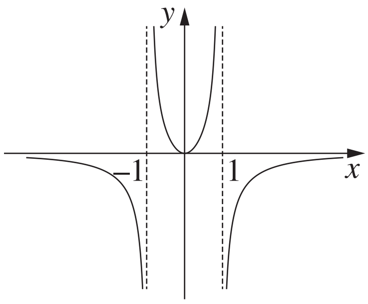
B. $-\frac{1}{2}$

C. $\frac{1}{2}$

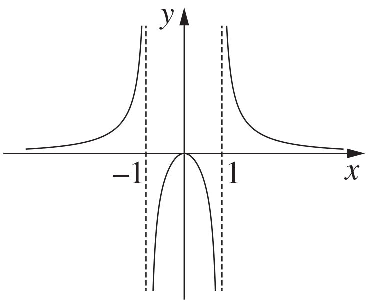
D. 2

5 Which graph best represents the function $y = \dfrac{2x^2}{1 - x^2}$?

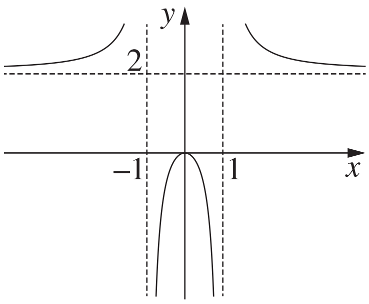
A.



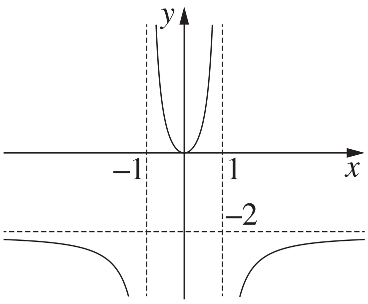
B.



C.



D.



6 The point $P\left(\dfrac{2}{p}, \dfrac{1}{p^2}\right)$, where $p \neq 0$, lies on the parabola $x^2 = 4y$.

What is the equation of the normal at P?

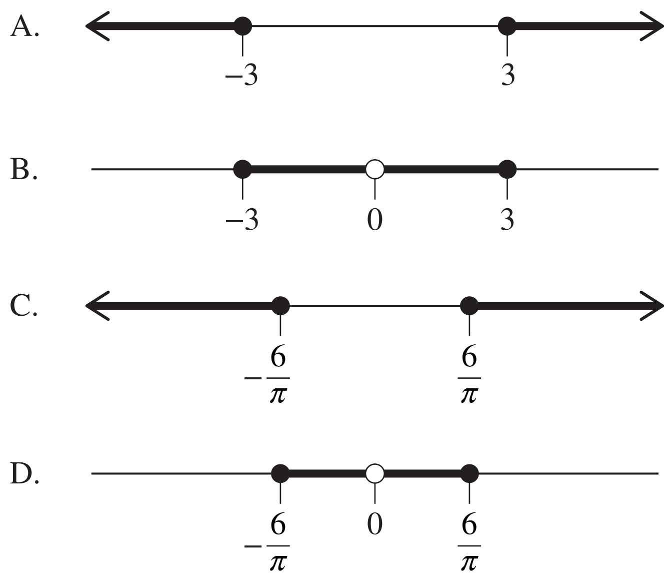
A. $py - x = -p$

B. $p^2y + px = -1$

C. $p^2y - p^3x = 1 - 2p^2$

D. $p^2y + p^3x = 1 + 2p^2$

7 Which diagram represents the domain of the function $f(x) = \sin^{-1}\left(\frac{3}{x}\right)$?



8 A stone drops into a pond, creating a circular ripple. The radius of the ripple increases from 0 cm, at a constant rate of 5 cm s^{-1}.

At what rate is the area enclosed within the ripple increasing when the radius is 15 cm?

A. 25π $cm^2\,s^{-1}$

B. 30π $cm^2\,s^{-1}$

C. 150π $cm^2\,s^{-1}$

D. 225π $cm^2\,s^{-1}$

9 When expanded, which expression has a non-zero constant term?

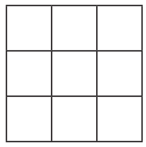
A. $\left(x + \frac{1}{x^2}\right)^7$

B. $\left(x^2 + \frac{1}{x^3}\right)^7$

C. $\left(x^3 + \frac{1}{x^4}\right)^7$

D. $\left(x^4 + \frac{1}{x^5}\right)^7$

10 Three squares are chosen at random from the 3×3 grid below, and a cross is placed in each chosen square.



What is the probability that all three crosses lie in the same row, column or diagonal?

A. $\frac{1}{28}$

B. $\frac{2}{21}$

C. $\frac{1}{3}$

D. $\frac{8}{9}$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) The point P divides the interval from $A(-4, -4)$ to $B(1, 6)$ internally in the ratio $2:3$. **1**

Find the x-coordinate of P.

(b) Differentiate $\tan^{-1}\left(x^3\right)$. **2**

(c) Solve $\dfrac{2x}{x+1} > 1$. **3**

(d) Sketch the graph of the function $y = 2\cos^{-1}x$. **2**

(e) Evaluate $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$, using the substitution $x = u^2 - 1$. **3**

(f) Find $\displaystyle\int \sin^2 x \cos x\,dx$. **1**

Question 11 continues on page 8

Question 11 (continued)

(g) The probability that a particular type of seedling produces red flowers is $\frac{1}{5}$. Eight of these seedlings are planted.

(i) Write an expression for the probability that exactly three of the eight seedlings produce red flowers. **1**

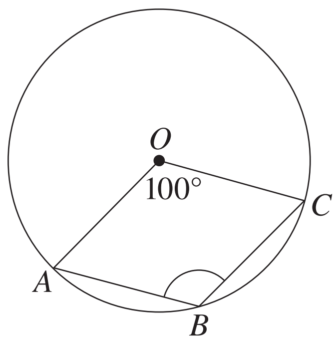
(ii) Write an expression for the probability that none of the eight seedlings produces red flowers. **1**

(iii) Write an expression for the probability that at least one of the eight seedlings produces red flowers. **1**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The points A, B and C lie on a circle with centre O, as shown in the diagram. The size of $\angle AOC$ is $100°$. **2**



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Find the size of $\angle ABC$, giving reasons.

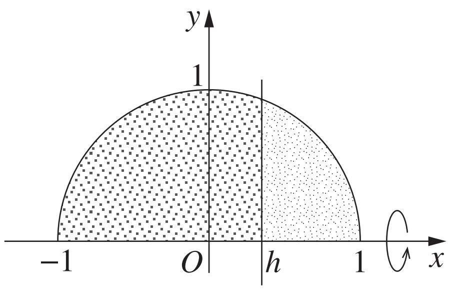
(b) (i) Carefully sketch the graphs of $y = |x+1|$ and $y = 3 - |x-2|$ on the same axes, showing all intercepts. **3**

(ii) Using the graphs from part (i), or otherwise, find the range of values of x for which **1**

$$|x+1| + |x-2| = 3.$$

Question 12 continues on page 10

(c) The region enclosed by the semicircle $y = \sqrt{1 - x^2}$ and the x-axis is to be divided into two pieces by the line $x = h$, where $0 \le h < 1$.



The two pieces are rotated about the x-axis to form solids of revolution. The value of h is chosen so that the volumes of the solids are in the ratio 2 : 1.

(i) Show that h satisfies the equation $3h^3 - 9h + 2 = 0$. **3**

(ii) Given $h_1 = 0$ as the first approximation for h, use one application of Newton's method to find a second approximation for h. **1**

(d) At time t the displacement, x, of a particle satisfies $t = 4 - e^{-2x}$. **3**

Find the acceleration of the particle as a function of x.

(e) Evaluate $\lim_{x \to 0} \dfrac{1 - \cos 2x}{x^2}$. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A particle is moving along the x-axis in simple harmonic motion centred at the origin. **3**

When $x = 2$ the velocity of the particle is 4.

When $x = 5$ the velocity of the particle is 3.

Find the period of the motion.

(b) Let n be a positive EVEN integer.

(i) Show that $(1+x)^n + (1-x)^n = 2\left[\binom{n}{0} + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right]$. **2**

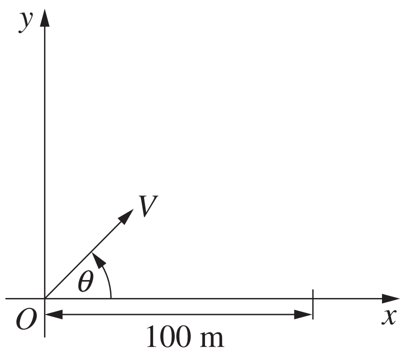
(ii) Hence show that **1**

$$n\left[(1+x)^{n-1} - (1-x)^{n-1}\right] = 2\left[2\binom{n}{2}x + 4\binom{n}{4}x^3 + \cdots + n\binom{n}{n}x^{n-1}\right].$$

(iii) Hence show that $\binom{n}{2} + 2\binom{n}{4} + 3\binom{n}{6} + \cdots + \frac{n}{2}\binom{n}{n} = n2^{n-3}$. **2**

Question 13 continues on page 12

(c) A golfer hits a golf ball with initial speed V m s^{-1} at an angle θ to the horizontal. The golf ball is hit from one side of a lake and must have a horizontal range of 100 m or more to avoid landing in the lake.



Neglecting the effects of air resistance, the equations describing the motion of the ball are

$$x = Vt\cos\theta$$
$$y = Vt\sin\theta - \frac{1}{2}gt^2,$$

where t is the time in seconds after the ball is hit and g is the acceleration due to gravity in m s^{-2}. Do NOT prove these equations.

(i) Show that the horizontal range of the golf ball is $\dfrac{V^2 \sin 2\theta}{g}$ metres. **2**

(ii) Show that if $V^2 < 100g$ then the horizontal range of the ball is less than 100 m. **1**

It is now given that $V^2 = 200g$ and that the horizontal range of the ball is 100 m or more.

(iii) Show that $\dfrac{\pi}{12} \le \theta \le \dfrac{5\pi}{12}$. **2**

(iv) Find the greatest height the ball can achieve. **2**

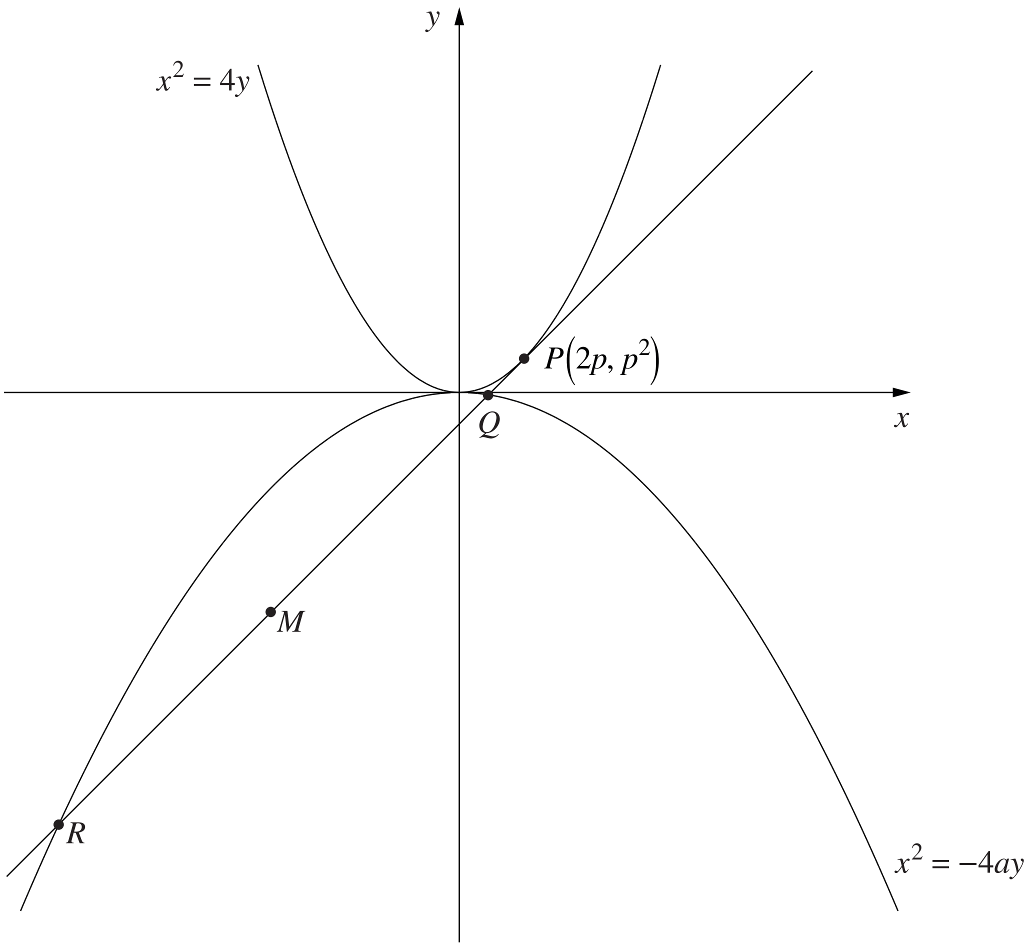
End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Prove by mathematical induction that $8^{2n+1} + 6^{2n-1}$ is divisible by 7, for any integer $n \geq 1$. **3**

(b) Let $P\left(2p, p^2\right)$ be a point on the parabola $x^2 = 4y$.

The tangent to the parabola at P meets the parabola $x^2 = -4ay$, $a > 0$, at Q and R. Let M be the midpoint of QR.



(i) Show that the x coordinates of R and Q satisfy **2**

$$x^2 + 4apx - 4ap^2 = 0.$$

(ii) Show that the coordinates of M are $\left(-2ap, -p^2(2a+1)\right)$. **2**

(iii) Find the value of a so that the point M always lies on the parabola $x^2 = -4y$. **2**

Question 14 continues on page 14

Question 14 (continued)

(c) The concentration of a drug in a body is $F(t)$, where t is the time in hours after the drug is taken.

Initially the concentration of the drug is zero. The rate of change of concentration of the drug is given by

$$F'(t) = 50e^{-0.5t} - 0.4F(t).$$

(i) By differentiating the product $F(t)e^{0.4t}$ show that **2**

$$\frac{d}{dt}\left(F(t)e^{0.4t}\right) = 50e^{-0.1t}.$$

(ii) Hence, or otherwise, show that $F(t) = 500\left(e^{-0.4t} - e^{-0.5t}\right)$. **2**

(iii) The concentration of the drug increases to a maximum. **2**

For what value of t does this maximum occur?

End of paper

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REFERENCE SHEET

– Mathematics –

– Mathematics Extension 1 –

– Mathematics Extension 2 –

Mathematics

Factorisation

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Angle sum of a polygon

$$S = (n-2) \times 180°$$

Equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Trigonometric ratios and identities

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

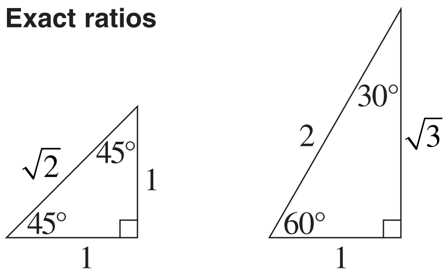
$$\sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

Exact ratios



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area of a triangle

$$\text{Area} = \frac{1}{2}ab\sin C$$

Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Perpendicular distance of a point from a line

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Slope (gradient) of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point–gradient form of the equation of a line

$$y - y_1 = m(x - x_1)$$

***n*th term of an arithmetic series**

$$T_n = a + (n-1)d$$

Sum to *n* terms of an arithmetic series

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

***n*th term of a geometric series**

$$T_n = ar^{n-1}$$

Sum to *n* terms of a geometric series

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

Limiting sum of a geometric series

$$S = \frac{a}{1 - r}$$

Compound interest

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\frac{dy}{dx} = F'(u)\frac{du}{dx}$

If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\frac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\frac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\frac{dy}{dx} = f'(x)\sec^2 f(x)$

Solution of a quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$$\alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$$

Equation of a parabola

$$(x-h)^2 = \pm 4a(y-k)$$

Integrals

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

Trapezoidal rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

Simpson's rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Logarithms – change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Angle measure

$180° = \pi$ radians

Length of an arc

$l = r\theta$

Area of a sector

$\text{Area} = \frac{1}{2}r^2\theta$

Mathematics Extension 1

Angle sum identities

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$$

t **formulae**

If $t = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2t}{1-t^2}$$

General solution of trigonometric equations

$\sin\theta = a, \qquad \theta = n\pi + (-1)^n \sin^{-1} a$

$\cos\theta = a, \qquad \theta = 2n\pi \pm \cos^{-1} a$

$\tan\theta = a, \qquad \theta = n\pi + \tan^{-1} a$

Division of an interval in a given ratio

$$\left(\frac{mx_2 + nx_1}{m+n},\ \frac{my_2 + ny_1}{m+n}\right)$$

Parametric representation of a parabola

For $x^2 = 4ay$,

$$x = 2at, \quad y = at^2$$

At $\left(2at,\ at^2\right)$,

tangent: $y = tx - at^2$

normal: $x + ty = at^3 + 2at$

At $\left(x_1,\ y_1\right)$,

tangent: $xx_1 = 2a\left(y + y_1\right)$

normal: $y - y_1 = -\frac{2a}{x_1}\left(x - x_1\right)$

Chord of contact from $\left(x_0,\ y_0\right)$: $xx_0 = 2a\left(y + y_0\right)$

Acceleration

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Simple harmonic motion

$$x = b + a\cos\left(nt + \alpha\right)$$

$$\ddot{x} = -n^2\left(x - b\right)$$

Further integrals

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Sum and product of roots of a cubic equation

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Estimation of roots of a polynomial equation

Newton's method

$$x_2 = x_1 - \frac{f\left(x_1\right)}{f'\left(x_1\right)}$$

Binomial theorem

$$\left(a+b\right)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$